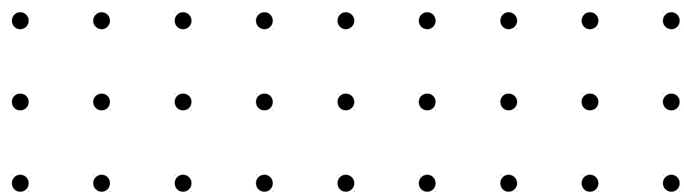
CS103 Handout 26
Spring 2017 May 9, 2017

Extra Practice Problems 4

Here's a set of practice problems you can work through to help prepare for the upcoming midterm exam. We'll release solutions next week.

Coloring a Grid

You are given a 3 × 9 grid of points, like the one shown below:



Suppose that you color each point in the grid either red or blue. Prove that no matter how you color those points, you can always find four points of the same color that form the corners of a rectangle.

Inductive Sets

A set S is called an ***inductive set*** if the follow two properties are true about S:

- $0 \in S$.
- For any number $x \in S$, the number $x + 1$ is also an element of S.

This question asks you to explore various properties of inductive sets.

i. Find two different examples of inductive sets.

ii. Prove that the intersection of any two inductive sets is also an inductive set.

iii. Prove that if S is an inductive set, then $\mathbb{N} \subseteq S$.

iv. Prove that $\mathbb{N}$ is the *only* inductive set that's a subset of all inductive sets. This proves that $\mathbb{N}$ is, in a sense, the most "fundamental" inductive set. In fact, in foundational mathematics, the set $\mathbb{N}$ is sometimes defined as the one inductive set that's a subset of all inductive sets.

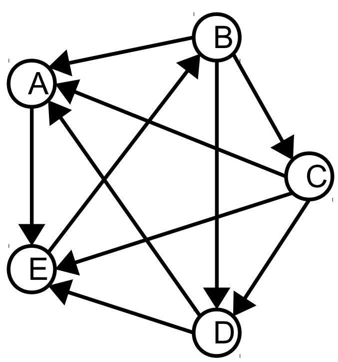
Tournament Graphs and Hamiltonian Paths

A ***tournament graph*** is a directed graph of n nodes where every pair of distinct nodes has exactly one edge between them. A ***Hamiltonian path*** is a path in a graph that passes through every node in a graph exactly once. Prove that every tournament graph has a Hamiltonian path. For the purposes of this problem you can consider the *empty path* of no nodes to be a Hamiltonian path through the empty graph.

Tournament Graphs and Binary Relations

This question explores the interaction between binary relations and tournaments.

Let's quickly refresh a definition. A ***tournament*** is a contest between some number of players in which each player plays each other player exactly once. We assume that no games end in a tie, so each game ends in a win for one of the players.



Here's a new definition to work with. If p is a player in tournament T, then we can define the set $W(p) = \{ x \mid x \text{ is a player in } T \text{ and } p \text{ beat } x \}$. Intuitively, $W(p)$ is the set of all the players that player p beat. For example, in the tournament on the left, $W(B) = \{A, C, D\}$.

Now, let's define a new binary relation. Let T be a tournament. We'll say that $p_1 \sqsubset_T p_2$ if $W(p_1) \subsetneq W(p_2)$. Intuitively, $p_1 \sqsubset_T p_2$ means that p_2 beat every player that p_1 beat, plus some additional players.

For example, in the tournament to the left, we have that $D \sqsubset_T C$ because $W(D) = \{A, E\}$ and $W(C) = \{A, D, E\}$. Similarly, we know $A \sqsubset_T D$ since $W(A) = \{E\}$ and $W(D) = \{A, E\}$.

Prove that if T is any tournament, then $\sqsubset_T$ is a strict order over the players in T.